Strongly Regular Graphs with (—1, 1, 0) Adjacency Matrix Having Eigenvalue 3

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1. INTRODUCTION

A graph is described by its adjacency matrix and hence it is provided with a spectrum. It is a general problem to confront the properties of a graph with the properties of its spectrum (cf. Hoffman [9]). In the present paper we consider ordinary graphs of finite order v with $(-1, 1, 0)$ adjacency matrix A satisfying

$$(A - \rho_1 I)(A - \rho_2 I) = (v - 1 + \rho_1\rho_2)J, \qquad \rho_1 > \rho_2, \tag{1}$$

$$AJ = \rho_0 J. \tag{2}$$

The only eigenvalues are ρ_0, ρ_1, ρ_2, with certain multiplicities. The numbers ρ_1 and ρ_2 turn out to be odd integers unless $\rho_1 + \rho_2 = 0$. Our main result will be that all such graphs are obtained for which $\rho_1 = 3$. These are the following:

(i) the graphs $H(n)$, the complements of the ladder graphs (cf. [9, 11]),

(ii) the lattice graphs $L_2(n)$, which Shrikhande [13] proved to be characterized by their parameters for $n \neq 4$,

(iii) the triangular graphs $T(n)$, which Chang [2] and Hoffman [8] proved to be characterized by their parameters for $n \neq 8$,

(iv) the exceptional graphs to the parameters of $L_2(4)$ and to the parameters of $T(8)$, due to Shrikhande [13] and Chang [3], respectively,

(v) the Petersen graph,

(vi) the Clebsch graph, which corresponds to the 16 lines on the Clebsch quartic surface (cf. Clebsch [14], Coxeter [5], Gewirtz [6]),

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(vii) the Schläfli graph, which corresponds to the 27 lines on a general cubic surface (cf. Coxeter [4]).

The graphs satisfying (1) and (2) are the strongly regular graphs of Bose [1]. Condition (2) means regularity. Graphs satisfying condition (1) are introduced in Section 2 under the name of strong graphs. They include graphs whose $(-1, 1, 0)$ adjacency matrix is orthogonal (cf. [7]). In Section 3 it is proved that strong graphs with $v - 1 + \rho_1\rho_2 \neq 0$ are regular.

Strong graphs with $(A - 3I)(A - \rho_2 I) = 0$ have been classified in [12]. In Section 4, as a consequence of a more general theorem on complete bipartite induced subgraphs, it is proved that most of the strongly regular graphs with $\rho_1 = 3$ contain no 3-claw. With this tool the proof of the standard form for the adjacency matrix of such graphs in Section 5, and of the final theorems in Section 6, mainly is a matter of elementary matrix multiplication.

As concepts of discrete mathematics strongly regular and strong graphs appear in geometry, engineering, statistics, and algebra; cf. [7, 11, 12] and the references cited therein.

2. STRONG GRAPHS

We consider undirected graphs of finite order v without loops and without multiple edges. A graph is described by the pair $\{V, A\}$ of the set V of its vertices and its adjacency matrix A defined by $A(x, y) = -1$ if $x \in V$ and $y \in V$ are adjacent, $A(x, y) = 1$ if $x \in V$ and $y \in V$ are nonadjacent, $A(x, x) = 0$ for all $x \in V$. There is an equivalence relation, generated by the operation of complementation, on the set of all graphs on v vertices. Here complementation with respect to any $x \in V$ means canceling all existing adjacencies for x and adding all nonexisting adjacencies for x, the effect on the adjacency matrix being multiplication by -1 of the row and the column corresponding to x (cf. [11, 12]).

For any vertices x and y with $A(x, y) = (-1)^h$ the integers

$$n_i(x) = |\{z : z \in V, A(x, z) = (-1)^i\}|, \qquad i = 1, 2,$$

$$p_{ij}^h(x, y) = |\{z : z \in V, A(x, z) = (-1)^i, A(y, z) = (-1)^j\}|, \qquad i, j = 1, 2,$$

are defined. So, for x and y adjacent $(h = 1)$ and nonadjacent $(h = 2)$, respectively, $p_{12}^h(x, y)$ is the number of vertices adjacent to x and nonadjacent to y, and $p_{21}^h(x, y)$ is the number of vertices nonadjacent to x

and adjacent to y. We shall be concerned with graphs with the property that both

$$p^1 = p^1_{12}(x, y) + p^1_{21}(x, y) \qquad \text{and} \qquad p^2 = p^2_{12}(x, y) + p^2_{21}(x, y)$$

are the same for all x, y, adjacent and nonadjacent, respectively.

DEFINITION. *A graph* $\{V, A\}$ *is* strong *if it is not void and not complete and if, for every* $h = 1, 2$, *there exists an integer* p^h *such that*

$$\forall x \in V, \forall y \in V, \qquad ((A(x, y) = (-1)^h) \Rightarrow (p^h_{12}(x, y) + p^h_{21}(x, y) = p^h)).$$

A graph is regular *if, for every* $i = 1, 2$, *the integer* $n_i(x)$ *is the same for all* $x \in V$. *A graph is* strongly regular [1] *if it is not void and not complete and if, for every* $h, i, j = 1, 2$, *the integer* $p^h_{ij}(x, y)$ *is the same for all* $x \in V$, $y \in V$ *with* $A(x, y) = (-1)^h$.

Trivially we have

THEOREM 1. *A graph is strongly regular if and only if it is strong and regular.*

The *complete bipartite graph* $K(\alpha, \beta)$, $\alpha + \beta = v$, $\alpha > 0$, $\beta > 0$, is the graph whose set of vertices V consists of two nonvoid disjoint subsets of orders α and β, each without adjacencies, whereas vertices belonging to different subsets are adjacent.

THEOREM 2. *For any strong graph, which is not* $K(\kappa, v - \kappa)$, $\kappa = 1, \ldots, v - 1$, *or its complement, the integers* p^1 *and* p^2 *are even.*

Proof. Let $\{V, A\}$ be nonvoid and noncomplete. Take $h = 1$ or $h = 2$. For any $x \in V, y \in V, x \neq y$, it follows from $p^h_{12}(x, y) + p^h_{21}(x, y) \equiv \varepsilon (\mathrm{mod}\ 2)$ that $n_1(x) \equiv n_2(y) + \varepsilon (\mathrm{mod}\ 2)$, for $\varepsilon = 0$ and for $\varepsilon = 1$. Now suppose that $\{V, A\}$ is strong with p^1 odd and p^2 even; then $A(x, y) = (-1)^\varepsilon$ implies $n_1(x) \equiv n_1(y) + \varepsilon (\mathrm{mod}\ 2)$ for $\varepsilon = 0$ and for $\varepsilon = 1$. Then $A(x, y) = A(y, z) = (-1)^\varepsilon$ implies $A(x, z) = 1$ for $\varepsilon = 0$ and for $\varepsilon = 1$. Hence $\{V, A\}$ is complete bipartite, which is excluded. Analogously, $p^1 \equiv 0 \ (\mathrm{mod}\ 2)$ and $p^2 \equiv 1 \ (\mathrm{mod}\ 2)$ lead to the complement of some $K(\kappa, v - \kappa)$. The case that both p^1 and p^2 are odd only occurs for $v = 2$ and is excluded. This proves the theorem.

The following theorems, the first of which is trivial, describe regular graphs and strong graphs in terms of their adjacency matrix. The next section will provide more details on the numbers ρ_0, ρ_1, ρ_2 to be introduced in these theorems.

THEOREM 3. *A graph $\{V, A\}$ is regular if and only if there exists an integer ρ_0 such that $AJ = \rho_0 J$.*

THEOREM 4. *A nonvoid and noncomplete graph $\{V, A\}$ is strong if and only if there exist real numbers ρ_1 and ρ_2 such that $\rho_1 > \rho_2$ and*

$$(A - \rho_1 I)(A - \rho_2 I) = (v - 1 + \rho_1\rho_2)J.$$

Proof. Let $\{V, A\}$ be nonvoid and noncomplete. Take $x \in V$, $y \in V$ with $A(x, y) = (-1)^h$ with $h = 1$ or $h = 2$. The element with indices x and y of the matrix $(A - \rho_1 I)(A - \rho_2 I)$ is the inner product of the rows

$$\begin{array}{cccccccc} -\rho_1 & (-1)^h & -\cdots- & -\cdots- & +\cdots+ & +\cdots+ & \text{of} & A - \rho_1 I, \\ (-1)^h & -\rho_2 & -\cdots- & +\cdots+ & -\cdots- & +\cdots+ & \text{of} & A - \rho_2 I, \end{array}$$

and equals

$$\begin{aligned} &-(-1)^h(\rho_1 + \rho_2) + p_{11}^h(x, y) - p_{12}^h(x, y) - p_{21}^h(x, y) + p_{22}^h(x, y) \\ &= v - 1 - (-1)^h(\rho_1 + \rho_2) - 2p_{12}^h(x, y) - 2p_{21}^h(x, y) - 1. \end{aligned}$$

Now let $\{V, A\}$ be strong. Take ρ_1 and ρ_2 such that

$$\rho_1 + \rho_2 = p^1 - p^2, \qquad -1 - \rho_1\rho_2 = p^1 + p^2, \qquad \rho_1 > \rho_2.$$

Then the inner product calculated above is independent of h, x, y and equals $v - 1 + \rho_1\rho_2$. Conversely, suppose we have

$$(A - \rho_1 I)(A - \rho_2 I) = (v - 1 + \rho_1\rho_2)J, \qquad \rho_1 > \rho_2$$

for real ρ_1, ρ_2. Then from

$$v - 1 - (-1)^h(\rho_1 + \rho_2) - 2p_{12}^h(x, y) - 2p_{21}^h(x, y) - 1 = v - 1 + \rho_1\rho_2,$$

for $h = 1, 2$ and for all x, y, it follows that

$$\begin{aligned} 2(p_{12}^1(x, y) + p_{21}^1(x, y)) &= (\rho_1 - 1)(1 - \rho_2), \\ 2(p_{12}^2(x, y) + p_{21}^2(x, y)) &= (\rho_1 + 1)(-1 - \rho_2) \end{aligned}$$

are independent of x and y, hence that the graph is strong.

3. CLASSIFICATION AND EXAMPLES

THEOREM 5. *Every strong graph* $\{V, A\}$ *with*

$$(A - \rho_1 I)(A - \rho_2 I) = (v - 1 + \rho_1\rho_2)J \neq 0$$

is regular, with $AJ = \rho_0 J$ *and* $(\rho_0 - \rho_1)(\rho_0 - \rho_2) = v(v - 1 + \rho_1\rho_2)$. *The spectrum of* A *consists of* ρ_1, ρ_2, *and* ρ_0, *which are odd integers and integer unless* $\rho_1 = -\rho_2 = \sqrt{v}$, $\rho_0 = 0$.

Proof. J is a linear combination of A^2, A, I. Hence these four matrices are simultaneously diagonalizable. $j = (1, 1, \ldots, 1)$ is an eigenvector of J belonging to the eigenvalue v, and hence of A belonging to the eigenvalue ρ_0, say. This implies

$$AJ = \rho_0 J, \qquad 1 - v < \rho_0 < v - 1, \qquad \rho_0 \text{ integer.}$$

Combination with the defining equation yields

$$(\rho_0 - \rho_1)(\rho_0 - \rho_2) = v(v - 1 + \rho_1\rho_2)$$

and proves that the only eigenvalues of A are ρ_0, ρ_1, ρ_2. Let μ_0, μ_1, μ_2 be their multiplicities; then $\mu_0 = 1$ and from $\operatorname{tr} A = 0$ it follows that

$$2\rho_0 + (v - 1)(\rho_1 + \rho_2) + (\mu_1 - \mu_2)(\rho_1 - \rho_2) = 0.$$

For $\mu_1 = \mu_2$ we have $\rho_0 = 0$, $\rho_1 = -\rho_2 = \sqrt{v}$. If $\mu_1 \neq \mu_2$ then ρ_1 and ρ_2 are rational, hence integral and odd.

THEOREM 6. *The spectrum of any strong graph* $\{V, A\}$ *with* $(A - \rho_1 I)(A - \rho_2 I) = 0$ *consists of* ρ_1 *and* ρ_2. *If* $\{V, A\}$ *is not* $K(\kappa, v - \kappa)$ *or its complement then* ρ_1 *and* ρ_2 *are odd integers unless* $\rho_1 = -\rho_2 = (v - 1)^{1/2}$.

Proof. The first statement is trivial. Let μ_i be the multiplicity of ρ_i, $i = 1, 2$; then from $\operatorname{tr} A = 0$ it follows that

$$v(\rho_1 + \rho_2) + (\mu_1 - \mu_2)(\rho_1 - \rho_2) = 0.$$

For $\mu_1 = \mu_2 = \frac{1}{2}v$ we have $\rho_1 = -\rho_2 = (v - 1)^{1/2}$ from $v - 1 + \rho_1\rho_2 = 0$. If $\mu_1 \neq \mu_2$ then ρ_1 and ρ_2 are rational, hence integral and, by Theorem 2, odd.

For the state of affairs concerning the existence and nonexistence of strong graphs $\{V, A\}$ with $A^2 = (v - 1)I$ and with $A^2 = vI - J$ we refer to [7]. Concerning the other strong graphs we first state some examples and a theorem, taken from [12].

Example 1. $H(n)$, $n > 3$, the *complement* of the *ladder graph* of order n, is the graph obtained from the complete graph on $2n$ vertices by deleting a 1-factor. This $H(n)$ is strongly regular with

$$v = 2n, \qquad \rho_0 = 3 - 2n, \qquad \rho_1 = 3, \qquad \rho_2 = -1.$$

Example 2. $L_2(n)$, $n > 1$, the *lattice graph* of order n, is the line graph of the complete bipartite graph $K(n, n)$. This $L_2(n)$ is strongly regular with

$$v = n^2, \qquad \rho_0 = (n-1)(n-3), \qquad \rho_1 = 3, \qquad \rho_2 = 3 - 2n,$$

and satisfies $v - 1 + \rho_1\rho_2 = 0$ only for $n = 2$ and for $n = 4$.

Example 3. $T(n)$, $n > 3$, the *triangular graph* of order n, is the line graph of the complete graph on n vertices. This $T(n)$ is strongly regular with

$$v = \tfrac{1}{2}n(n-1), \qquad \rho_0 = \tfrac{1}{2}(n-2)(n-7), \qquad \rho_1 = 3, \qquad \rho_2 = 7 - 2n,$$

and satisfies $v - 1 + \rho_1\rho_2 = 0$ only for $n = 5$ and $n = 8$.

Example 4. The *Petersen graph* is the graph whose vertices are the ten unordered pairs out of five symbols, adjacency between any two pairs being defined if and only if they have no common symbol. The Petersen graph, whose complement is the Desargues graph $T(5)$, is strongly regular with

$$v = 10, \qquad \rho_1 = \rho_0 = 3, \qquad \rho_2 = -3.$$

Example 5. $L_2'(4)$, the *pseudolattice graph*, is the graph obtained from $L_2(4)$ by complementation with respect to the vertices of any subgraph which is an 8-circuit. In addition, $L_2'(4)$ may be defined as the complement of the net $(4, 3)$ which corresponds to a nonextendable latin square of order 4. This $L_2'(4)$ is strongly regular with

$$v = 16, \qquad \rho_1 = \rho_0 = 3, \qquad \rho_2 = -5,$$

and was proved by Shrikhande [13] to be the only such graph apart from $L_2(4)$.

Example 6. $T'(8)$, $T''(8)$, $T'''(8)$, the three *pseudotriangular graphs*, are defined as follows.

$T'(8)$ is obtained from $T(8)$ by complementation with respect to any four independent vertices.

$T''(8)$ is obtained from $T(8)$ by complementation with respect to the eight vertices of any subgraph consisting of a 3-circuit and a 5-circuit without adjacencies between them.

$T'''(8)$ is obtained from $T(8)$ by complementation with respect to the 12 vertices of any subgraph which is the line graph of the 8-circuit with four antipodal adjacencies

These $T'(8)$, $T''(8)$, $T'''(8)$ are strongly regular with

$$v = 28, \qquad \rho_1 = \rho_0 = 3, \qquad \rho_2 = -9,$$

and were proved by Chang [3] to be the only such graphs apart from $T(8)$.

THEOREM 7. *Strong graphs with* $(A - 3I)(A - \rho_2 I) = 0$ *only exist for* $\rho_2 = -1, -3, -5, -9$. *Any such graph is equivalent to* $L_2(2)$, $T(5)$, $L_2(4)$, $T(8)$, *respectively.*

For the proof of this theorem we refer to [12]. The following theorem is a consequence of the results of Shrikhande and of Chang. It also could be proved on the basis of Theorem 7.

THEOREM 8. *The only strongly regular graphs with* $(A - 3I)(A - \rho_2 I) = 0$, $AJ = 3J$, *are* $L_2(4)$, $T(8)$, *the Petersen graph*, $L_2'(4)$, $T'(8)$, $T''(8)$, $T'''(8)$.

All graphs mentioned in Theorem 8, except for $L_2(4)$, $T(8)$, have 3-claws $K(3, 1)$. It will be proved in the following section that these five are the only strongly regular graphs with $\rho_1 = 3$ having this property.

4. COMPLETE BIPARTITE INDUCED SUBGRAPHS

Let $\{V, A\}$ be a strong graph with defining equation

$$(A - \rho_1 I)(A - \rho_2 I) = (v - 1 + \rho_1\rho_2)J.$$

Let $K(\alpha, \beta)$ be an induced subgraph of $\{V, A\}$. For $\beta = 1$ this is an α-claw.

THEOREM 9. *A necessary condition for the existence of a sub-$K(\alpha, \beta)$ in a strong graph $\{V, A\}$ is*

$$\alpha + \beta \leqslant \tfrac{1}{4}(\rho_1 - 3)(3 - \rho_2) + 4.$$

Proof. Take $x \in K(\alpha, \beta)$ and $y \in K(\alpha, \beta)$ with $A(x, y) = -1$. Suppose $\alpha > 1$. Arrange the elements of V in such a way that the first $\alpha + 1$ rows of A are

$$\begin{array}{cccccccc}
0 & - & +\cdots+ & -\cdots- & +\cdots+ & -\cdots- & -\cdots- & +\cdots+ \\
- & 0 & -\cdots- & +\cdots+ & -\cdots- & +\cdots+ & -\cdots- & +\cdots+ \\
j & -j & J-I & -J & B & C & D & E,
\end{array}$$

where

$$1,\ 1,\ \alpha - 1,\ \beta - 1,\ p_{21}^1 - \alpha + 1,\ p_{12}^1 - \beta + 1,\ p_{11}^1,\ p_{22}^1$$

are the consecutive numbers of columns. We apply the defining equation to the third (block) row with the first and with the second column:

$$-\rho_2 J + J + (J - I - \rho_1 I)J + JJ + BJ - CJ - DJ + EJ$$
$$= (v - 1 + \rho_1\rho_2)J,$$
$$-J + \rho_2 J - (J - I - \rho_1 I)J - JJ - BJ + CJ - DJ + EJ$$
$$= (v - 1 + \rho_1\rho_2)J.$$

Hence

$$J(-\rho_1 - \rho_2 - 2 + \alpha + \beta) = CJ - BJ$$

and

$$-\rho_1 - \rho_2 - 2 + \alpha + \beta \leqslant \tfrac{1}{2}(\rho_1 - 1)(1 - \rho_2) + 2 - \alpha - \beta,$$

since the right-hand side is the total number of columns of the matrix $(B \quad C)$. From this the inequality follows.

Remark. In the theorem the equality sign

$$\alpha + \beta = \tfrac{1}{4}(\rho_1 - 3)(3 - \rho_2) + 4$$

holds if and only if $B = -J$ and $C = J$.

THEOREM 10. *A necessary condition for the existence of a sub-*$K(\alpha, \beta)$ *with*

$$\alpha \geqslant 3, \qquad \beta \geqslant 1, \qquad \alpha + \beta = \tfrac{1}{4}(\rho_1 - 3)(3 - \rho_2) + 4$$

in a strongly regular graph $\{V, A\}$ *is*

$$|1 + \rho_0 + 2\beta - 2\alpha| \leqslant v - 1 + \rho_1\rho_2.$$

Proof. Referring to the proof of and the remark to Theorem 9, we consider any pair of vertices x', x'' of the block. By $A(x', x'') = 1$, $B = -J$, $C = J$ we have

$$\beta - \alpha + 1 + p^1_{21} \leqslant p^2_{11}, \qquad \alpha - \beta - 1 + p^1_{12} \leqslant p^2_{22}.$$

From this the inequality follows.

THEOREM 11. *The only strong graphs with* $\rho_1 = 3$, $\rho_2 = -1$ *are* $K(3, 1)$, $L_2(2)$, $T(4)$, $H(n)$, $n = 3, 4, \ldots$.

Proof. Let $\{V, A\}$ be strong with $\rho_1 = 3$, $\rho_2 = -1$. For $v = 4$ only $K(3, 1)$ and $K(2, 2) = L_2(2)$ apply. For $v > 4$ we have $v - 1 + \rho_1\rho_2 \neq 0$; hence, by Theorem 5, $\{V, A\}$ is strongly regular with

$$(\rho_0 - 3)(\rho_0 + 1) = v(v - 4), \qquad 1 - v < \rho_0 < v - 1.$$

This implies

$$v = 3 - \rho_0, \qquad p^1_{12} = 1, \qquad p^2_{22} = p^2_{12} = 0,$$

which, as the only possibility, leaves $T(4)$ and $H(n)$.

THEOREM 12. *Strongly regular graphs with* $\rho_1 = 3$, $\rho_0 \neq 3$ *contain no 3-claws.*

Proof. Application of Theorems 9 and 10 in the case $\rho_1 = 3$ yields $\alpha \leqslant 3$ for the order of any α-claw $K(\alpha, 1)$, whereas for $\alpha = 3$

$$|\rho_0 - 3| \leqslant v - 1 + 3\rho_2$$

is necessary. Combination with

$$(\rho_0 - 3)(\rho_0 - \rho_2) = v(v - 1 + 3\rho_2)$$

yields, by $\rho_0 \neq 3$, as a necessary condition for $\alpha = 3$

$$|\rho_0 - 3| \leqslant |\rho_1 - \rho_2| - 1 + 3\rho_2,$$

whence $\rho_2 = -1$. However, by Theorem 11, this admits only graphs which have no 3-claws. Therefore, $\alpha = 3$ is not possible.

5. THE STANDARD ADJACENCY MATRIX FOR $\rho_1 = 3$, $\rho_0 \neq 3$

We shall derive a standard form for the adjacency matrix A of a strongly regular graph $\{V, A\}$ with defining equations

$$(A - 3I)(A - \rho_2 I) = (v - 1 + 3\rho_2)J, \qquad AJ = \rho_0 J, \qquad \rho_0 \neq 3.$$

By Theorem 12 we know that there are no 3-claws. We shall divide V into eleven disjoint subsets, four of which consist of solely one vertex:

$$V = \{p\} \cup \{q\} \cup \{r\} \cup \{s\} \cup X' \cup X'' \cup Y' \cup Y'' \cup Z' \cup Z'' \cup T.$$

Take any $q \in V, r \in V$ with $A(q, r) = -1$. Take any $p \in V$ with $A(p, q) = 1$, $A(p, r) = -1$. Define

$$\bar{Z} = \{z \in V : A(z, p) = A(z, q) = -1\},$$

$$Z = \{z \in V : A(z, p) = A(z, q) = A(z, r) = -1\},$$

$$T = \{t \in V : A(t, p) = A(t, q) = 1\}.$$

Since $\{V, A\}$ has no 3-claws there are no adjacencies of r to the vertices of T. Considering p and r we learn that r is adjacent to $p_{11}^1 - |Z|$ vertices which are adjacent to p and nonadjacent to q. Again, from q and r we see that r is adjacent to $p_{11}^1 - |Z|$ vertices which are nonadjacent to p and adjacent to q. We conclude that r is adjacent to

$$n_1 = 2 + |Z| + 2p_{11}^1 - 2|Z|$$

other vertices. Expressing n_1 and p_{11}^1 in v and the eigenvalues (cf. [12, p. 190]) we obtain $|Z| = \frac{1}{2}(v - \rho_0 + 2\rho_2 - 3)$. By $|\bar{Z}| = p_{11}^2 = \frac{1}{2}(v - \rho_0 + 2\rho_2 + 1)$ we have $|\bar{Z}| = |Z| + 2$. Hence r is nonadjacent to only one vertex of $\bar{Z}$, s say. By application of the above reasoning to the other vertices of $\bar{Z}$ it follows that any $z' \in \bar{Z}$ is nonadjacent to only one $z'' \in \bar{Z}$, that $\bar{Z} = Z \cup \{r\} \cup \{s\}$, and that Z can be split into two subsets Z', Z'' of equal order such that

$$A(Z', Z'') = \begin{pmatrix} I - J & 2I - J \\ 2I - J & I - J \end{pmatrix}.$$

We define the following subsets of V:

$$X' = \{x \in V: A(x,p) = -1, A(x,r) = 1\} \setminus \{s\},$$

$$X'' = \{x \in V: A(x,p) = 1, \quad A(x,r) = -1\} \setminus \{q\},$$

$$Y' = \{y \in V: A(y,q) = 1, \quad A(y,r) = -1\} \setminus \{p\},$$

$$Y'' = \{y \in V: A(y,q) = -1, A(y,r) = 1\} \setminus \{s\}.$$

These sets all have order $p_{12}^1 - 1 = -\frac{1}{2}(1 + \rho_2)$ and are mutually disjoint. From the regularity of A and from the absence of 3-claws the rest of the adjacencies of p, q, r, s with the vertices of $X', X'', Y', Y'', Z', Z'', T$ is readily found. From their orders it may be seen that these eleven disjoint subsets fill up V completely. Now it will be shown that the adjacency matrix A may be given the following form:

$$\begin{array}{ccccccccccc}
0 & + & - & - & -j^T & j^T & -j^T & j^T & -j^T & -j^T & j^T \\
+ & 0 & - & - & j^T & -j^T & j^T & -j^T & -j^T & -j^T & j^T \\
- & - & 0 & + & j^T & -j^T & -j^T & j^T & -j^T & -j^T & j^T \\
- & - & + & 0 & -j^T & j^T & j^T & -j^T & -j^T & -j^T & j^T \\
-j & j & j & -j & I-J & J-2I & A_{12} & A_{12} & A_{13} & -A_{13} & A_{14} \\
j & -j & -j & j & J-2I & I-J & A_{12} & A_{12} & A_{13} & -A_{13} & A_{14} \\
-j & j & -j & j & A_{12}^T & A_{12}^T & I-J & J-2I & A_{23} & -A_{23} & A_{24} \\
j & -j & j & -j & A_{12}^T & A_{12}^T & J-2I & I-J & A_{23} & -A_{23} & A_{24} \\
-j & -j & -j & -j & A_{13}^T & A_{13}^T & A_{23}^T & A_{23}^T & I-J & 2I-J & J \\
-j & -j & -j & -j & -A_{13}^T & -A_{13}^T & -A_{23}^T & -A_{23}^T & 2I-J & I-J & J \\
j & j & j & j & A_{14}^T & A_{14}^T & A_{24}^T & A_{24}^T & J^T & J^T & A_{44}
\end{array}$$

The corresponding sets and their orders are

$$\begin{array}{ccccccccccc}
p, & q, & r, & s, & X', & X'', & Y', & Y'', & Z', & Z'', & T, \\
1, & 1, & 1, & 1, & \multicolumn{6}{c}{-\frac{1}{2}(1+\rho_2) \text{ each,}} & \text{and}
\end{array}$$

$$|Z'| = |Z''| = \tfrac{1}{4}(v - \rho_0 + 2\rho_2 - 3), \qquad |T| = \tfrac{1}{2}(v + \rho_0 + 2\rho_2 - 1).$$

In order to prove this we first remark that from the absence of 3-claws it follows that

$$A(X', X') = A(X'', X'') = A(Y', Y') = A(Y'', Y'') = I - J.$$

The defining equation, written for p and r with X' and with X'', yields

$$JA(X', X'') = JA^T(X', X'') = \tfrac{1}{2}J(-5 - \rho_2).$$

Since $A(X', X'')$ has order $-\frac{1}{2}(1 + \rho_2)$, each of its rows and columns has only one element -1. Suitable arrangement of the vertices of X'' yields $A(X', X'') = J - 2I$. Analogously we obtain $A(Y', Y'') = J - 2I$. Second, we wish to prove

$$A(X', Y' \cup Y'' \cup Z' \cup Z'' \cup T) = A(X'', Y' \cup Y'' \cup Z' \cup Z'' \cup T).$$

Denote the left-hand side by A_1 and the right-hand side by A_2. The defining equation, written for X' with X'', yields

$$A_1 A_2^T = -2(1 + \rho_2)I + (v - 1 + 3\rho_2)J.$$

Since both the diagonal elements of $A_1 A_2^T$ and the number of columns of A_1 and of A_2 equal $v + \rho_2 - 3$, it follows that $A_1 = A_2$. Analogously we have

$$A(Y', X' \cup X'' \cup Z' \cup Z'' \cup T) = A(Y'', X' \cup X'' \cup Z' \cup Z'' \cup T).$$

Third, we prove

$$A(Z', X' \cup X'' \cup Y' \cup Y'') = -A(Z'', X' \cup X'' \cup Y' \cup Y'').$$

Denote the left-hand side by A_3 and the right-hand side by $-A_4$. The defining equation, written for Z' with Z'', yields

$$A_3 A_4^T = 2(1 + \rho_2)I,$$

which implies $A_3 = -A_4$. Now we may conclude that the adjacency matrix A of $\{V, A\}$ has the form announced above.

Finally some relations between the submatrices of A are listed in the order of their application in the following section. The defining equation, written for p with X', and the regularity condition $AJ = \rho_0 J$, written for X', yield

$$A_{14}J = (v - 1 + 3\rho_2)J, \qquad A_{12}J = \tfrac{1}{2}(-v + 2 - 3\rho_2 + \rho_0)J.$$

Analogously we have

$$A_{24}J = (v - 1 + 3\rho_2)J, \qquad A_{12}^TJ = \tfrac{1}{2}(-v + 2 - 3\rho_2 + \rho_0)J.$$

The regularity condition, written for Z', yields

$$JA_{13} + JA_{23} = 0.$$

The defining equation, written for X' with Z', yields

$$2A_{12}A_{23} = (5 + \rho_2)A_{13}$$

and, written for p with T,

$$JA_{44} = \tfrac{1}{2}(3v - \rho_0 + 10\rho_2 + 3)J.$$

From the defining equation, written for X' with X', Y' with Y', Z' with Z', X' with Y', respectively, we obtain

$$2A_{12}A_{12}^T + 2A_{13}A_{13}^T + A_{14}A_{14}^T = -2(\rho_2 + 1)I + (v - 1 + 3\rho_2)J,$$

$$2A_{12}^TA_{12} + 2A_{23}A_{23}^T + A_{24}A_{24}^T = -2(\rho_2 + 1)I + (v - 1 + 3\rho_2)J,$$

$$A_{13}^TA_{13} + A_{23}^TA_{23} = -(\rho_2 + 1)I,$$

$$-(5 + \rho_2)A_{12} + 2A_{13}A_{23}^T + A_{14}A_{24}^T = J(v - 1 + 3\rho_2).$$

6. STRONGLY REGULAR GRAPHS WITH $\rho_1 = 3$

From Theorem 8 we know all strongly regular graphs with $\rho_1 = \rho_0 = 3$. Now let $\{V, A\}$ be strongly regular with $\rho_1 = 3$, $\rho_0 \neq 3$. We use the standard form for A derived in the preceding section.

1. If $\rho_2 = -1$ then, by Theorem 11, we have $L_2(2)$, $T(4)$, $H(n)$. From now on we assume $\rho_2 \neq -1$.

2. If $v - \rho_0 + 2\rho_2 - 3 = 0$, that is, if $Z' = Z'' = \phi$, then

$$(\rho_0 - 3)(\rho_0 - \rho_2) = v(v - 1 + 3\rho_2) = (\rho_0 - 2\rho_2 + 3)(\rho_0 + \rho_2 + 2),$$

whence $4\rho_0 = \rho_2{}^2 + 2\rho_2 - 3$. Put

$$\rho_2 = 3 - 2n, \quad n \geqslant 3; \qquad \text{then } \rho_0 = n^2 - 4n + 3, \quad v = n^2, \quad n \neq 4.$$

In addition we have

$$|X'| = |X''| = |Y'| = |Y''| = n - 2, \qquad |T| = (n-2)^2, \qquad A_{12} = J.$$

It readily follows that in this case $\{V, A\}$ is precisely the lattice graph $L_2(n)$, $n \neq 2, 4$. Shrikhande [13], by other methods, was the first to prove this result, that is, to characterize $L_2(n)$ for $n \neq 4$ in terms of its parameters.

3. If $v - \rho_0 + 2\rho_2 - 3 = 4$, that is, if $|Z'| = |Z''| = 1$, then

$$(\rho_0 - 3)(\rho_0 - \rho_2) = (\rho_0 - 2\rho_2 + 7)(\rho_0 + \rho_2 + 6),$$

whence $8\rho_0 = \rho_2{}^2 + 4\rho_2 - 21$. Put

$$\rho_2 = 7 - 2n, \quad n \geqslant 5;$$

$$\text{then } \rho_0 = \tfrac{1}{2}(n^2 - 9n + 14), \quad v = \tfrac{1}{2}n(n-1), \quad n \neq 8.$$

In addition we have

$$|X'| = |X''| = |Y'| = |Y''| = n - 4, \qquad |T| = \tfrac{1}{2}(n-4)(n-5),$$

$$A_{12} = J - I, \qquad A_{13} = A_{23} = -j.$$

It readily follows that in this case $\{V, A\}$ is precisely the triangular graph $T(n)$, $n \neq 4, 8$. Chang [2] and Hoffman [8], in partial results preceded by others, were the first to prove this result, that is, to characterize $T(n)$ for $n \neq 8$ in terms of its parameters.

4. From now on we assume $|Z'| = |Z''| \geqslant 2$, that is,

$$v - 1 + 3\rho_2 \geqslant \rho_0 + \rho_2 + 10.$$

Under this assumption the formulas

$$JA_{12} = \tfrac{1}{2}(-v + 2 - 3\rho_2 + \rho_0)J, \qquad 2A_{12}A_{23} = (5 + \rho_2)A_{13},$$

$$JA_{13} + JA_{23} = 0,$$

which were obtained in Section 5, imply

$$JA_{13} = JA_{23} = 0.$$

We shall prove $JA_{12} \leqslant 0$ and then $\rho_2 \leqslant \rho_0 < 3$. Take any $z' \in Z'$, $z'' \in Z''$ with $A(z', z'') = 1$. Arrange the vertices of X' and of Y' in such a way that

$$A(z', X') = A(z', Y') = (j^T \quad -j^T),$$
$$A(z'', X') = A(z'', Y') = (-j^T \quad j^T),$$

with $j^T = (1, \ldots, 1)$ of length $-\frac{1}{4}(1 + \rho_2)$. From the absence of 3-claws originating from p we infer that

$$A_{12} = \begin{pmatrix} -J & B \\ C & -J \end{pmatrix}$$

for certain square B and C of order $-\frac{1}{4}(1 + \rho_2)$, whence

$$JA_{12} = \tfrac{1}{2}(-v + 2 - 3\rho_2 + \rho_0)J \leqslant 0.$$

In view of the order of A_{12} this implies

$$1 + \rho_2 \leqslant -v + 2 - 3\rho_2 + \rho_0 \leqslant 0,$$

$$\rho_0 - \rho_2 \geqslant v - 1 + 3\rho_2 \geqslant \rho_0 + 1 > \rho_0 - 3,$$

$$v(\rho_0 - \rho_2) \geqslant (\rho_0 - 3)(\rho_0 - \rho_2) > v(\rho_0 - 3),$$

whence $\rho_2 \leqslant \rho_0 < 3$. Indeed, from $\rho_0 < \rho_2$ and the first inequality it would follow that $v \leqslant \rho_0 - 3$, and from $\rho_0 > 3$ and the second inequality that $v < \rho_0 - \rho_2$. However, both conclusions contradict $-v + 3 - 3\rho_2 + \rho_0 \leqslant 0$.

5. If $\rho_0 = \rho_2$ then $v = 1 - 3\rho_2$ and $JA_{12} = \frac{1}{2}(1 + \rho_2)J$, whence $A_{12} = -J$. As a consequence, from $JA_{23} = 0$, $2A_{12}A_{23} = (5 + \rho_2)A_{13}$, we have $\rho_2 = \rho_0 = -5$, $v = 16$ and

$$|X'| = |X''| = |Y'| = |Y''| = |Z'| = |Z''| = 2, \qquad |T| = 0.$$

From the last formulas of Section 5 we obtain

$$A_{13}A_{13}^T = A_{23}A_{23}^T = 4J - 2I, \qquad A_{13}^T A_{13} + A_{23}^T A_{23} = 4I, \qquad A_{13}A_{23}^T = 0.$$

These equations are satisfied by

$$A_{13} = \begin{pmatrix} + & + \\ - & - \end{pmatrix}, \qquad A_{23} = \begin{pmatrix} + & - \\ - & + \end{pmatrix}.$$

The resulting graph $\{V, A\}$ is unique* since interchange of X', X'' and

* S. S. Shrikhande informed the author about his (independent) proof, to be published in *Sankhya*, of the uniqueness of this graph and of the nonexistence of such graphs with $v = 28$, $\rho_1 = 3$, $\rho_2 = \rho_0 = -9$.

Y', Y'' has the effect of interchanging A_{13} and A_{23} within A. The complement $\{V, -A\}$ of the resulting graph consists of two subgraphs, a 5-claw and a Petersen graph on 5 symbols, whose adjacencies are described by inclusion, if the end vertices of the 5-claw are taken as the symbols of the Petersen graph (cf. Gewirtz [6]). In terms of polytopes the 16 vertices and 80 adjacencies of the graph $\{V, A\}$ can be identified with the 16 vertices and 80 edges of the polytope $h\gamma_5$, also denoted by 1_{21} (cf. [4, pp. 158, 201]). This remark is due to H. S. M. Coxeter, who also points out the relation of this polytope to the 16 lines (and 80 pairs of skew lines) on Clebsch's quartic surface (cf. Clebsch [14]). Therefore, $\{V, A\}$ will be called the *Clebsch graph.*

6. From now on we assume $\rho_0 \neq \rho_2$. Then $\rho_2 < \rho_0 < 3$, $v - 1 + 3\rho_2 < 0$, $|T| > 0$. From

$$2JA_{44} = (3v - \rho_0 + 10\rho_2 + 3)J,$$

$$-v - \rho_0 - 2\rho_2 + 3 \leqslant 3v - \rho_0 + 10\rho_2 + 3$$

we have $-1 \leqslant v - 1 + 3\rho_2$, whence $v + 3\rho_2 = 0$, $A_{44} = I - J$. $\rho_0 = 3 + \rho_2$ is, since $\rho_0 = 0$ contradicts $A_{12}J \leqslant 0$, a consequence of

$$v(v - 1 + 3\rho_2) = (\rho_0 - 3)(\rho_0 - \rho_2).$$

Now we are able to combine the formulas $(A - 3I)(A - \rho_2 I) = -J$, $AJ = (3 + \rho_2)J$ into

$$\begin{pmatrix} -3 & j^T \\ j & A - 3I \end{pmatrix}\begin{pmatrix} -\rho_2 & j^T \\ j & A - \rho_2 I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Application of Theorem 7 yields $\rho_2 = -1, -3, -5, -9$. However, in view of our earlier assumptions, only $\rho_2 = -9$ is relevant. Therefore, we have $v = 27$, $\rho_1 = 3$, $\rho_2 = -9$, $\rho_0 = -6$,

$$|X'| = |X''| = |Y'| = |Y''| = 4, \quad |Z'| = |Z''| = 3, \qquad |T| = 1.$$

From [12, p. 194], it follows that the resulting graph $\{V, A\}$ is unique. Now there is a well-known graph which meets our conditions and which, therefore, is the final $\{V, A\}$ we are looking for. This is the graph whose vertices are the 27 lines on a general cubic surface, adjacencies being defined if and only if two lines do not intersect. We shall refer to this graph as the *Schläfli graph* after its earliest describer (cf. Coxeter [4, p. 211]). This graph also may be defined by the 27 vertices and the edges of Gosset's six-dimensional polytope 2_{21}. The Schläfli graph may be obtained from $T(8)$ by complementation with respect to the 12 vertices which are adjacent to any one vertex, and then suppression of that vertex. Notice that the graph has as its subgraph a $H(5)$ on $p \cup q \cup r \cup s \cup Z' \cup Z''$ and a pseudolattice graph on $X' \cup X'' \cup Y' \cup Y''$. Hoffman and Ray-Chaudhuri (cf. [7]) observed the Schläfli graph to be the example of the greatest valency ($n_1 = 16$) for a regular connected graph G with $\lambda(G) = -2$, $G \neq H(n)$, to be not a line graph. Here $\lambda(G) = -2$ is the least eigenvalue of the (1, 0) adjacency matrix of G, corresponding to our $\rho_1 = 3$.

Summarizing the results of the present section we have

THEOREM 13. *The only strongly regular graphs with $\rho_1 = 3$, $\rho_0 \neq 3$ are $H(n)$, $L_2(n)$ for $n \neq 4$, $T(n)$ for $n \neq 8$, the Clebsch, and the Schläfli graph.*

Combination of Theorems 8 and 13 yields

THEOREM 14. *The only strongly regular graphs with $\rho_1 = 3$ are the complements of the ladder graphs, the lattice graphs, the triangular graphs, the pseudolattice graph, the pseudotriangular graphs, and the graphs of Petersen, Clebsch, and Schläfli.*

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